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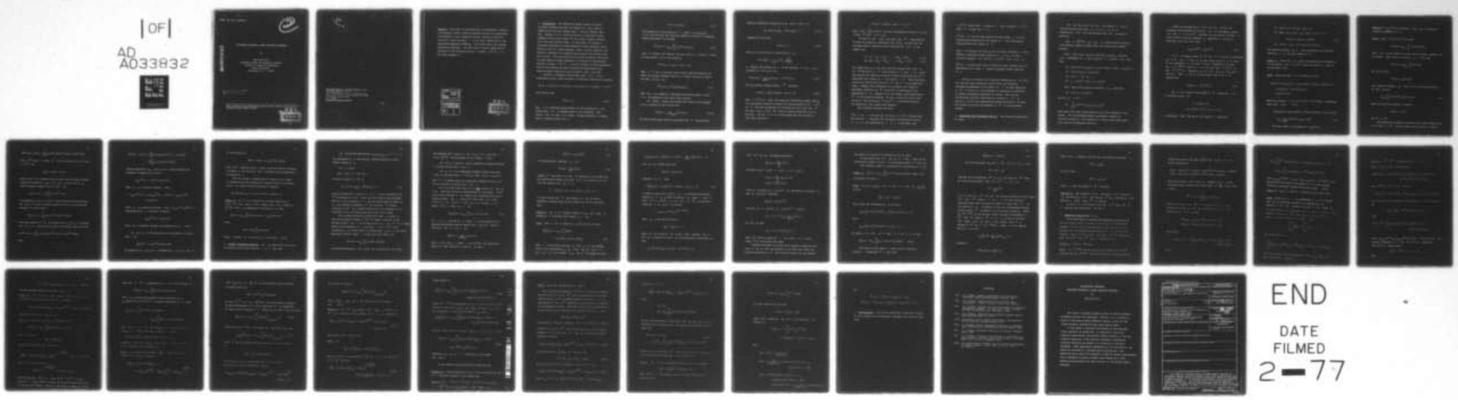

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PERTURBED STOCHASTIC LINEAR REGULATOR PROBLEMS+

by

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Abstract. This paper is concerned with the approximate solution of stochastic optimal control problems which arise by perturbing the stochastic linear regulator problem, through an additive term with a small parameter δ in the drift coefficient of the unperturbed dynamical equations. The system states are assumed completely observable. Our main results concern expansions of solutions of the perturbed equation in powers $\delta, \delta^2, \delta^3, \ldots$ of the small parameter δ.

1. Introduction. The problem of optimal control of Markov diffusion processes has been the subject of a great deal of research over the past several years. See for instance [FR1], [B1]. However, it is a difficult matter to calculate optimal feedback control laws, except for the linear regulator problem and a few other special cases. In this paper, we consider a nonlinear perturbation of the stochastic linear regulator, which takes the form of a small quantity times a certain function, and develop a technique for computing approximately the optimal feedback control. The system states are assumed completely observable. Our main results concern expansions of solution of the perturbed problem in powers $\delta, \delta^2, \delta^3, \ldots$ and the validity of these expansions. Part of this problem has been considered by Kolmanovskii but under strongest conditions than we consider here. See [Ko1]

Consider a stochastic system whose state $\xi^\delta(t)$ is an n dimensional vector, which satisfies a stochastic differential equation

$$d\xi^\delta(t) = (A(t)\xi^\delta(t) + \delta g(\xi^\delta(t)) + B(t)u(t))dt + \sigma(t)dw(t) \qquad (1.1^\delta)$$

with initial data

$$\xi^\delta(s) = x. \qquad (1.2^\delta)$$

Here w is a brownian motion process of some dimension d. The system state $\xi(t)$ is assumed known to the controller. The control $u(t)$ at time t is a vector, of some dimension k, chosen using a feedback control law Y:

$$u(t) = Y(t,\xi(t)). \tag{1.3}$$

The problem is to find among all $Y \in \mathscr{Y}(R^t)$, to be defined in §2, one for which the following quadratic criterion of expected system performance is minimum.

$$J^{\delta}(s,x;u) = E_{s,x} \int_s^T L(t,\xi^{\delta}(t),u(t))dt \tag{1.4^{δ}}$$

where T denotes the terminal time and $L(t,x,u) = x'M(t)x + u'N(t)u$. For convenience, we use the notations

$$f^{\delta,u}(t,x) = A(t)x + \delta g(x) + Bu .$$

When $\delta = 0$, this is the well known linear regulator problem, for which the optimal feedback control is a linear function of state. See [FR1, Section 6.5].

$$Y^{o*}(s,x) = -N^{-1}(s)B'(s)K(s)x. \tag{1.5}$$

Here $K(s)$ is a symmetric, non-negative definite matrix of size $n \times n$ and bounded on any finite time interval.

Let $\phi^{\delta}(s,x)$ denote the minimum cost function and consider it as a function of the initial data

$$\phi^{\delta}(s,x) = \inf_{Y \in \mathscr{Y}(R^k)} J^{\delta}(s,x;Y). \tag{1.6^{δ}}$$

We like to show under certain conditions that ϕ^{δ} satisfies the partial differential equation for all $(s,x) \in [0,T] \times R^n$

$$\phi_s^\delta + \frac{1}{2}\,\mathrm{tr}\{\sigma\sigma'\phi_{xx}^\delta\} + H^\delta(s,x,\phi_x^\delta) = 0 \qquad (1.7^\delta)$$

together with the data

$$\phi^\delta(T,x) = 0 \qquad (1.8^\delta)$$

where tr is the trace of a square matrix, i.e.,

$$\mathrm{tr}\{\sigma\sigma'\phi_{xx}^\delta\} = \sum_{i,j=1} (\sigma\sigma')_{ij} \frac{\partial^2\phi^\delta}{\partial x_i \partial x_j},$$

ϕ_x denotes the gradient of ϕ in the variables $x = (x_1,\ldots,x_n)'$, regarded as a row vector and

$$H^\delta(s,x,P) = \min_{u\varepsilon K=R^k} [L(s,x,u) + P\cdot f^{\delta,u}(s,x)] \qquad (1.9^\delta)$$

and the optimal feedback control $Y^{\delta*}$ satisfies

$$y'N(s)y + \phi_x^\delta(s,x)\cdot B(s)y = \min \text{ on } R^k \qquad (1.10)$$

when $y = Y^{\delta*}(s,x)$. Thus, the completely observable optimal problem is in principle reduced to solving the Cauchy problem (1.7^δ) - (1.8^δ) for ϕ^δ and then minimizing the left-hand side of (1.10) over R^k for each $(s,x) \in [0,T] \times R^n$. This is usually difficult to do in practice. But for $\delta = 0$, it is well-known that the solution is (see [FR1], Section 5)

$$\phi^{o}(s,x) = x'K(s)x + q(s), \quad 0 \leq s \leq T$$

where $q(s) = \int_s^T tr\{\sigma\sigma' K\} dt$ and the corresponding solution of (1.10) is just $Y^{o*}(s,x)$ as in (1.5).

We wish to find ϕ, ϕ_x^δ (and hence also $Y^{\delta *}$) approximately in terms of quantities computable from ϕ^o, ϕ_x^o. We seek that the following type of expansions hold uniformly for (s,x) is any compact set:

$$\phi^\delta = \phi^o + \delta\theta_1 + \delta^2\theta_i + \ldots + \delta^k\theta_k + o(\delta^k), \tag{1.11}$$

$$\phi_x^s = \phi_x^o + \delta\theta_{1x} + \delta^i\theta_{2x} + \ldots + \delta^k\theta_{kx} + o(\delta^k). \tag{1.12}$$

The coefficients in (1.11) must have the property that $k!\theta_k$ is the k^{th} derivative of ϕ^δ with respect to δ when $\delta = 0$. Hence they satisfy the equations found by formally differentiating (1.7^δ) repeatedly with respect to δ and setting $\delta = 0$. These equations involve the partial derivatives of H^δ and ϕ^o of corresponding orders. Whether such expansions are available will depend on smoothness properties of H^δ which will be guaranteed by the assumptions in §2. Suppose we use the optimal unperturbed policy Y^{o*} in the perturbed problem. We like to know how close to the optimum is the performance $J^\delta(s,x;Y^{o*})$ in perturbed problems. Our method will also answer this equation.

The following notations are used

$C^\ell(\Gamma)$ — If Γ is an open set, we write $g \in C^\ell(\Gamma)$ to mean that the function g together with its partial derivatives of orders $j = 1,\ldots,\ell$ are continuous on Γ. If Γ is not open, then

$g \in C^{\ell}(\Gamma)$ means that g agrees on Γ with a function $h \in C^{\ell}(\Gamma')$ where Γ' is open and $\Gamma \subset \Gamma'$.

$C^{1,2}(Q)$ — It has the same meaning as above except g is twice continuously differential with respect to x and continuously differentiable with respect to t.

$C_p(Q)$ — It denotes the class of all continuous functions ψ which satisfy a polynomial growth condition on Q, i.e., for some positive constants c,m, $|\psi(t,x)| \leq c(1+|x|^m)$ when $(t,x) \in Q$.

$C_p^{(\ell)}(Q)$ — It means the class of functions which together with its derivatives up to order ℓ satisfy polynomial growth condition on Q.

We begin in section 2 by discussing assumptions on $f, L, \mathscr{Y}(k)$ and then get some preliminary results about the existence, uniqueness, boundedness of the moments of ξ^{δ} and some properties of H^{δ}. In section 3, the existenee and uniqueness of solutions of dynamic programming equations are proved. Then we use a verification theorem to show that the solution is ϕ^{δ}. In section 4, we give the approximation method and prove it is valid and finally we discuss the goodness of Y^{o*} in the perturbed problem.

2. Assumptions and Preliminary Results. The following assumptions are made.

(AI) $A(t), B(t), M(t)$ and $N(t)$ are bounded C^∞ matrix-valued functions with size $n \times n$, $n \times k$, $n \times n$, $k \times k$ respectively. $M(t)$ is a semi-positive and $N(t)$ is positive definite.

(AII) $g \in C_p^{(\ell)}(R^n)$ and $g_x(x)$ is a matrix-valued function with diagonal elements bounded above and off-diagonal elements bounded.

(AIII) There exist positive constants M_1, M_2, M_3, constants α_1, α_2 independent of δ, and a positive C^∞ function $V(x)$ such that

(a) $\frac{1}{2}\,\mathrm{tr}\{\sigma(t)\sigma'(t)V_{xx}(x)\} + V_x(x)\cdot(A(t)x + \delta g(x)) \leq M_1(1+V(x))$

(b) $(1+|x|)|V_x(x)| \leq M_1(1+V(x))$

(c) $V(x) \to \infty$ as $|x| \to \infty$

(d) $\alpha_1 + M_2|x|^2 \leq V(x) \leq \alpha_2 + M_3|x|^2$

(AIV) There exist positive constants c_1, c_2 such that

(a) $|\sigma(t,x)| \leq c_1(1+|x|)$

and for all $\nu \in R^n$

(b) $\sum_{i,j=1}^{n} (\sigma(t)\sigma'(t))_{ij}\nu_i\nu_j \geq c_2|\nu|^2$.

This means that noise enters directly into each component of the system. The corresponding dynamic programming equation is uniformly parabolic. This enables us to apply result about parabolic partial differential equation.

Under the assumptions of (AIII) and (AV). We have the existence and uniqueness of solution $\eta(t)$ of free-control system, i.e., let $u = 0$ in (1.1^{δ}). Moreover, for any positive integer m, there exists a positive constant C_m depending on t, M_1, M_2, M_3 such that, see [F3], [K1], [K2]

$$E_{s,x}|\eta(t)|^m \leq C_m(1+|x|^m) \tag{2.1}$$

If K is compact subset of R^k, then H^{δ} is clearly well-defined by (1.9^{δ}). Let $u = V(s,x,p)$ be the unique vector in K at which H^{δ} is a minimum on K for every (s,x,p) belonging to an open set Γ of R^{2n+1}. Then if Γ is a set such that $V \in C^{\ell}(\Gamma)$, we have $H^{\delta} \in C^{\ell+1}(\Gamma)$. Moreover, $H_s = L_s + Pf_s$, $H_x = L_x + Pf_x$, $H_p^s = f$, see [F1]. When $k = R^k$, we can see easily that H^{δ} is a C^{∞} function and

$$V(s,x,P) = -\frac{1}{2} N^{-1}B'P'. \tag{2.2}$$

Let K be a closed convex subset of R^k containing 0 as an interior point. Let

$$Q = [0,T] \times R^n$$

$$Q_r = [0,T] \times \{|x| \leq r\}, \ r = 1,2,\ldots$$

we define by $\mathscr{Y}(K)$ the set of all function Y such that

(a) $Y(s,x) \in K$ for all $(s,x) \in Q$

(b) When $(s,x),(s,y) \in Q_r$ with $0 \leq s \leq T' < T$

$$|Y(s,x) - Y(s,y)| \leq \alpha_r |x-y| \tag{2.3}$$

(c) For all $(s,x) \in Q$, $|Y(s,x)| \leq \beta(1+|x|)$

The positive consants α_r, β may be different for different functions Y. α_r may also depend on T'.

<u>Lemma 2.1</u>. Conditions (2.3) insure the existence and uniqueness of the process $\xi^\delta(t)$ in (1.1^δ), given the control Y and the initial data (1.2^δ).

<u>Proof</u>. Using the same V as in (AIII), we have

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\{\sigma(t)\sigma'(t)V_{xx}(x)\} + V_x(x)\cdot(A(t)x + \delta g(x) + B(t)Y(t,x)) \\
&\leq M_1(1+V(x)) + \beta(1+|x|)|V_x(x)| \\
&\leq M_1(1+\beta)(1+V).
\end{aligned}$$

Hence the solution ξ^δ of (1.1^δ) with (1.2^δ) exists. Uniqueness comes from (2.3b). Q.E.D.

Furthermore, there exists a positive constant C_m' such that

$$E_{s,x} \operatorname*{Max}_{s\leq t\leq 1} |\xi^\delta(t)|^m \leq C_m'(1 + |x|^m). \tag{2.4}$$

The next lemma is an estimate on $E_{sx}|\xi^\delta(t)|$.

Lemma 2.2. $E_{s,x}|\xi^{\delta}(t)| \leq C''_m(1+|x|)$ where C''_m is a positive constant independent of K.

Proof: Since $0 \in K$, by (1.6^{δ}) we have

$$\phi^{\delta}(s,x) \leq E_{s,x} \int_s^T \eta'(t)M(t)\eta(t)dt$$

where $\eta(t)$ is the solution of free control system. Because $M(t)$ is bounded. There exists a constant $\beta_1 > 0$ such that

$$\phi^{\delta}(s,x) \leq \beta_1 E_{s,x} \int_s^T |\eta(t)|^2 dt.$$

By (2.1) we have

$$\phi^{\delta}(s,x) \leq \beta_2(1+|x|^2)$$

for a positive constant β_2. From (1.6^{δ}) and positive definite of $N(t)$ we have

$$E_{s,x} \int_s^T |u^{\delta *}(t)|^2 dt \leq \frac{\beta_2}{\gamma}(1+|x|^2) \tag{2.5}$$

where the positive constant γ satisfies

$$\mu' N(t)\mu \geq \gamma|\mu|^2$$

for all $\mu \in R^k$.

Now subtract the equation governing free control system from (1.1^{δ}) with $u = u^{\delta *}$. Using the mean value theorem, we obtain

$$d(\xi^{\delta*}-\eta)(t) = (A(t) + \delta \int_0^1 g_x(\eta+\lambda(\xi^{\delta*}-\eta))d\lambda)(\xi^{\delta*}-\eta)(t)dt + B(t)u^{\delta*}(t)dt$$

with $(\xi^{\delta*}-\eta)(s) = 0$ where $\xi^{\delta*}$ is the solution of (1.1^{δ}) with $u = u^{\delta*}$. Let

$$g_x(x) = G_1(x) + G_2(x)$$

where $G_1(x)$ is a diagonal matrix-valued function whose diagonal elements are those of $g_x(x)$ and $G_2(x)$ is the same as g_x except diagonal elements are all zero. Let

$$A_1(t) = A(t) + \delta \int_0^1 G_2(\eta+\lambda(\xi^{\delta*}-\eta))d\lambda.$$

By assumptions, $A_1(t)$ is still a bounded matrix-valued function. Let $X_2(t,v)$ be the principal matrix solution of the following equation at initial time v,

$$dX_1(t,v) = \delta \int_0^1 G_1(\eta+\lambda(\xi^{\delta*}-\eta))d\lambda\, X_1(t,v)dt.$$

Since the elements of G_1 are bounded above, $X_1(t,u)$ is bounded for $s \leq v \leq t$. Using the variation of constants formula, we get

$$(\xi^{\delta*}-\eta)(t) = \int_s^T X_1(t,v)[A_1(v)(\xi^{\delta*}-\eta)(v) + B(v)u^{\delta*}(v)]dv.$$

Then

$$|\xi^{\delta*}(t) - \eta(t)|^2 \leq 2\left|\int_s^T X_1(t,v)A_1(v)(\xi^{\delta*}(v) - \eta(v))dv\right|^2$$

$$+ 2\left|\int_s^T X_1(t,v)B(v)u^{\delta*}(v)dv\right|^2.$$

Taking expectation $E_{s,x}$ and by (2.5), Cauchy-Schwartz and Gronwall's inequalities, we have

$$E_{s,x}|\xi^{\delta*}(t) - \eta(t)|^2 \leq \beta_3(1+|x|^2)$$

where β_3 is a positive constant. Hence

$$E_{s,x}|\xi^{\delta*}(t)|^2 \leq 2\ E_{s,x}|\xi^{\delta*}(t) - \eta(t)|^2 + 2\ E_{s,x}|\eta(t)|^2$$

$$\leq \beta_4(1+|x|^2)$$

where β_4 is a positive constant. Since $(E_{s,x}|\xi^{\delta*}(t)|^p)^{1/p}$ is nondecreasing as p increases, we obtain

$$E_{s,x}|\xi^{\delta*}(t)| \leq C''_m(1+|x|)$$

where C''_m is positive constant not depending on K. Q.E.D.

Let $X_2(t,v)$ be the principal matrix solution at initial time v of

$$dX_2(t,v) = \delta\ G_1(\xi^{\delta*}(t))X_2(t,v)dt.$$

By assumption on $G_1, X_1(t,v)$ is bounded for $s \leq v \leq t$. Let W be the solution of

$$dW(t) = (A(t) + \delta\, g_x(\xi^{\delta *}(t)))W(t)dt$$

with $W(s)$ = identity matrix. Again, using the similar technique as before, we can show that $W(t)$ is bounded and the bound does not depend on x.

This next lemma, a modification of Lemma V.5.2 of [FR1], is concerned with the probabilistic representation for solution $\psi(s,x)$ of a linear partial differential equation

$$\psi_s + \frac{1}{2}\,\mathrm{tr}\{\sigma\sigma'\psi_{xx}\} + \psi_x\cdot f + g(s,x)\psi + h(s,x) = 0.$$

Lemma 2.3. Let ψ be a solution of the above equation in $[0,T] \times R^n$ with $\psi(T,x) = \Psi(x)$, suppose that ψ, h, Ψ belong to $C_p^{1,2}(Q)$ and g is bounded and continuous on Q, then

$$\psi(s,x) = E_{s,x}\int_s^T D(u)h(u,\xi(u)du + E_{s,x}D(T)\Psi(\xi(T))$$

where

$$D(u) = \exp\int_s^u g(v,\xi(v))dv$$

Proof. Consider ψD in the proof of cited lemma. Q.E.D.

3. Dynamic Programming Equation. Let $\mathcal{F}_0$ denote the set of all non-negative real valued function ψ on Q such that

(i) the partial derivatives $\psi_s, \psi_{x_i}, \psi_{x_i, x_j}$, $i,j = 1,\ldots,n$ are continuous on Q and satisfy a Hölder condition on each compact subset of Q.

(ii) $\psi \in C_p(Q)$

(iii) $\psi(T,x) = 0$ for all x.

We seek a solution in $\mathscr{F}_0$ of

$$\psi_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'\psi_{xx}\} + H^\delta(s,x,\psi_x) = 0 \tag{3.1}$$

with the Cauchy data $\psi(T,x) = 0$. If ϕ^δ is such a solution, let $Y^{\delta*}$ be defined by (1.10). Since the first term on the left-hand side of (1.10) is quaratic in y and the second term is linear in y, (1.10) uniquely determines $Y^{\delta*}$. The function $Y^{\delta*}$ clearly satisfies (2.3a). By a similar proof as for (F.2 Theorem 2.2) it satisfies (2.3b), we shall prove later that (2.3c) holds.

The following theorem is quoted from [FR1]. It tells us that the existence of each ϕ^δ and $Y^{\delta*}$ imply a solution to the minimum problem. Let Γ be an open subset of Q and $\partial^*\Gamma$ be a closed subset of the boundary of Γ such that $(\tau,\xi^\delta(\tau)) \in \partial^*\Gamma$ with probability 1, for every choice of initial data $(s,x) \in \Gamma$ and every admissible control, where τ is the first exit time. Let

$$J(s,x;Y) = E_{s,x}\int_s^\tau L(t,\xi^\delta(t),u(t))dt.$$

[Verification Theorem]: Let $\psi(s,x)$ be a solution of (3.1) with the boundary data $\psi(s,x) = 0$ for $(s,x) \in \partial^*\Gamma$ such that ψ is in $C_p^{1,2}(\Gamma)$ and continuous on the closure $\bar{\Gamma}$, then

(a) $\psi(s,x) \leq J(s,x;Y)$ for any admissible feedback control Y and any initial data $(s,x) \in \Gamma$.

(b) If Y^* is an admissible feedback control such that (1.10) is satisfied when $Y = Y^*(s,x)$, then $\psi(s,x) = J(s,x;Y^*)$.

This Y^* is optimal among all admissible feedback control laws, for all choices of initial data $(s,x) \in \Gamma$.

Let us now show that there is a unique solution in $\mathcal{F}_0$ of (3.1). This will be done by approximation in two stages. In the first step we assume K is a compact set containing zero as an interior point. Let τ_r denote the first time $t \leq T$ when $|\xi^\delta(t)| = r$, if $|\xi^\delta(t)| < r$ for $s \leq t \leq T$, we set $\tau_r = T$, where $\xi^\delta(s) = x$ and $|x| < r$. Let

$$J_r^\delta(s,x;Y) = E_{s,x}\int_s^{\tau_r} L(t,\xi^\delta(t),u(t))dt \tag{3.2}$$

As $r \to \infty$, τ_r increase to T. Since L is non-negative, the monotone convergence theorem implies that $J_r(s,x;Y)$ tends to $J^\delta(s,x;Y)$. For $r = 1,2,\ldots,$ let

$$\phi_r^\delta(s,x) = \min_{Y \in \mathcal{Y}(k)} J_r(s,x;Y),$$

then $0 \leq \phi_1^\delta \leq \phi_2^\delta \leq \ldots$, since $0 \in K$, we have $\phi_r^\delta \leq J_r^\delta(s,x;0)$. From (2.1) and $J_r^\delta(s,x;0) \leq J(s,x;0)$ we have

$$\phi_r^\delta(s,x) \le \nu_1(1+|x|^2) \tag{3.3}$$

for some positive constants ν_1. Let

$$\phi^\delta(s,x) = \lim_{r\to\infty} \phi_r^\delta(s,x) \tag{3.4}$$

Clearly, ϕ^δ satisfies (3.3) too. By Theorem VI 6.1 of [FR1] and the verification theorem, for each r, ϕ_r^δ is a solution of (3.1) with the boundary data $\phi_r^\delta = 0$ on

$$\Sigma_r = ([0,T] \times \{|x| = r\}) \cup (\{T\} \times \{|x| \le r\}).$$

In order to show that ϕ^δ also belongs to $\mathscr{F}_0$, we need to establish a uniform bound on any compact set for the gradients $(\phi_r^\delta)_x$.

<u>Lemma 3.1</u>. Let B be a compact subset of Q_{r_0}, then $(\phi_r^\delta)_x$ is bounded on B uniformly with respect to $r > r_0$.

<u>Proof</u>. With $\phi = \phi_r^\delta$ in Lemma 5.3, p. 494 of [F1], we have

$$(\phi_r^\delta)_x(s,x) = E_{s,x}\int_s^{\tau_r} L_x W dt +$$

$$+ E_{s,x}(\phi_r^\delta)_x(\tau_r, \xi^\delta(\tau_r))W(\tau_r) \tag{3.5}$$

where τ_r is the exit time from Q_r with $Y = Y^*$, the optimal control law corresponding to ϕ_r^δ. Since $(\phi_r^\delta)_x(T;\xi^\delta(T)) = 0$, $|L_x| \le \alpha_1 L + \alpha_2$ for suitable α_1, α_2 and W is bounded, we have

$$|(\phi_r)_x(s,x)| \le \alpha_1\phi_r^\delta(s,x) + \alpha_2(T-s) + \max_{|x|=r} |(\phi_r^\delta)_x| P\{\tau_r < T\}.$$

Let N_r be a number such that

$$\phi_r^\delta(s,x) \le N_r(r-|x|)$$

whenever $|x| < r$. Then

$$|(\phi_r^\delta)_x(s,x)| \le \alpha_1\phi_r^\delta(s,x) + \alpha_2(T-s) + N_r P\{\tau_r < T\}. \qquad (3.6)$$

In order to show that $N_r P_r\{\tau_r < T\}$ is uniformly bounded with respect to $r > r_0$, we have to estimate N_r. Given x take x^o with $|x^o| = r$, $|x-x^o| = r - |x|$. Let $\nu = -\frac{x^o}{r}$, we construct a barrier θ at (s,x^o) as follows

$$\theta(s,x) = e^{T-S}(1-e^{-k_r\nu\cdot(x-x^o)})$$

where k_r is the positive root of

$$c_r k^2 - M_r k - 1 = 0$$

where M_r is a bound of $|At + \delta g(x) + BY^*|$ whenever $|x| \le r$ and c_2 is defined in (AIV). By straightforward calculation, we have

$$\theta_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'\theta_{xx}\} + \theta_x\cdot(Ax + \delta g + BY^*) \le -1$$

and $\theta \geq 0$ on Q_r. By maximum principle,

$$\phi_r^\delta(s,x) \leq (\operatorname*{Max}_{Q_r} L^{Y*})\theta.$$

Moreover, since $\theta(s,x^o) = 0$ and $r - |x| = |x-x^o|$,

$$\theta(s,x) \leq (\operatorname*{Max}_{Q_r} |\theta_x|)(r-|x|)$$

$$\theta(s,x) \leq e^T k_r (r-|x|).$$

Since K is compact, $k_r \leq D_1(1+r)^\ell$ for some positive constant D_1 and $L^Y \leq D_2(1+r)^2$, therefore

$$\phi_r^\delta(s,x) \leq D_3(1+r)^{2+\ell}$$

for some $D_3 > 0$. We take $N_r = D_3(1+r)^{2+\ell}$. Finally,

$$P\{\tau_r < T\} \leq r^{-\lambda} E_{s,x} \operatorname*{Max}_{s \leq t \leq T} |\xi^\delta(t)|^\lambda.$$

By (2.4), we have

$$P\{\tau_r < T\} \leq r^{-\lambda} C''_\lambda (1+|x|^\lambda),$$

where C''_λ does not depend on r. If we take $\lambda \geq 2 + \ell$ and recall (3.3), this proves the lemma.

Standard estimates for second order parabolic equations (see [Fr1], P. 60, 65, 191) and passages to the limit then imply the desired properties of ϕ^δ. The technical details of the argument are similar to the proof of Theorem VI.6.2, of [FR1].

We have shown that $\phi^\delta \in \mathcal{F}_0$ and $Y^* \in \mathcal{Y}(k)$. Hence by the verification theorem, $\phi^\delta(s,x)$ is the minimum values of $J^\delta(s,x;Y)$.

The following lemma is a probabilistic representation of ϕ^δ_x.

Lemma 3.2. $\phi^\delta_x(s,x) = E_{s,x}\int_s^T L_x(t,\xi^{\delta*}(t),u^*(t))W(t)dt$, where $W(t)$ is defined in Section 2.

Proof. By (3.6), $N_r R_r(\tau_r < T) \to 0$ are $r \to \infty$ and $\phi^\delta_r = \lim_{r\to\infty} \phi^\delta_{rx}$, we have

$$|\phi^\delta_x| \le \alpha_1\phi^\delta + \alpha_2(T-S).$$

Thus, using the boundedness of W, we obtain

$$E_{s,x}|\phi^\delta_x(\tau^r,\xi^\delta(\tau^r))W(\tau^r)| \le \text{const.}(1+r)^2 P\ (\tau_r < T).$$

Hence

$$E_{s,x}|\phi^\delta_x(\tau^r,\xi^\delta(\tau^r))W(\tau^r)| \to 0 \quad \text{as} \quad r \to \infty.$$

By Lemma 5.3 of [F1] on ϕ^δ and $\tau^r \to T$ as $r \to \infty$, we have

$$\phi^\delta_x(s,x) = E_{s,x}\int_s^T L_x(t,\xi^{\delta*}(t),u^*(t))W(t)dt. \qquad \text{Q.E.D.}$$

From Lemma 3.2 and Lemma 2.2, there exists a positive constant ν independent of K such that

$$|\phi_x^\delta(s,x)| \le \nu(1+|x|). \tag{3.7}$$

Let us now consider the case $K = R^k$, For $m = 1,2,\ldots$, let

$$K^m = \{|Y| \le m\}.$$

Consider the corresponding $H^{\delta m}$ in (3.1) and solution $\phi^{\delta m}$ found by the previous method. Then $\phi^{\delta 1} \ge \phi^{\delta 2} > \ldots \ge 0$. Let

$$\phi^\delta = \lim_{m\to\infty} \phi^{\delta m}.$$

By (3.3) and (3.7), $\phi^{\delta m}$ and $\phi_x^{\delta m}$ are uniformly bounded on each compact set. Since $H^{\delta m}$ tends to H^δ as $m \to \infty$, the same reasoning indicated right after the proof of Lemma 3.1 shows that $\phi^\delta \in \mathscr{F}_0$ and satisfies (3.1). It remains to show the corresponding optimal control policy $Y^{\delta *}$ satisfies (2.3c) and hence belongs to $\mathscr{Y}(R^k)$. Let $Y^m = Y^{\delta m*}$ be the optimal control function corresponding to $\phi^{\delta m}$. Then Y^m tends to $Y^{\delta *}$ as $m \to \infty$. We want to estimate $|Y^m|$. Given (s,x), let $M(y) = y'Ny + \phi_x^\delta \cdot By$. Then M is minimum of K^m for $Y = Y^m = Y^m(s,x)$. Since 0 is an interior point of K, we have

$$M_y(Y^m)\cdot Y^m = \frac{d}{dz} M(zY^m)\big|_{z=1} \le 0.$$

Therefore,

$$2Y^{m'}N(t)Y^m + \phi_x^\delta \cdot BY^m \le 0.$$

Since $B(t)$ is bounded, we have for some positive constants ν_2

$$|Y^m| \leq \nu_2 |\phi_x^\delta|$$

By (3.5) then

$$|Y^m| \leq \nu_3(1+|x|)$$

where ν_3 does not depend on K^m. Therefore

Theorem 3.1. The function $\phi^\delta(s,x)$ defined by (1.6^δ) belongs to $\mathcal{F}_0$ and satisfies (3.1). The function $Y^{\delta*}(s,x)$ defined by (1.10) belongs to $\mathcal{Y}(R^k)$. Thus $Y^{\delta*}$ is optimal.

Actually ϕ^δ is as smooth as we want (C^∞), since H^δ is C^∞ and also the Cauchy data, See [Fr1].

4. Asymptotic Formulas for $\phi^\delta, \phi_x^\delta$

We are now ready to consider the expansions of solution of the perturbed problem in terms of the solution of unperturbed problem. At the end of this section we also indicate how the methods tell the goodness of the policy Y^{o*} in the perturbed problem. Since $A(t), B(t), M(t)$ and $N(t)$ are C^∞ functions, then $\phi^\delta, \phi^o, Y^{\delta*}$ and Y^{o*} are C^∞ functions too.

Lemma 4.1. $\phi^\delta(s,x) \to \phi^o(s,x)$.

Proof. Let $Y^{\delta*}, Y^{o*}$ be the controls corresponding to ϕ^δ, ϕ^o respectively and $\xi^{\delta*}, \xi^{o*}$ be the corresponding Markov processes respectively (given the same initial data (s,x)). Let ξ,ζ be the solutions of

$$d\xi(t) = (A(t)\xi(t) + \delta g(\xi(t)) + B(t)Y^{o*}(t,\xi^{o*}(t)))dt + \sigma(t)dw(t)$$
$$d\zeta(t) = (A(t)\zeta(t) + B(t)Y^{\delta*}(t,\xi^{\delta*}(t)))dt + \sigma(t)dw(t)$$

with initial data $\xi(s) = \zeta(s) = x$. Suppose $X(t,v)$ is the principal matrix solution at initial time v of $\frac{dX}{dt} = A(t)X$, then we have

$$\xi^{\delta*} - \zeta = \delta \int_s^t X(t,v)g(\xi^{\delta*}(v))dv$$

then it is easy to see $\xi^{\delta*} \to \zeta$ in probability as $\delta \to 0$. Similarly we have $\xi \to \xi^{o*}$ in probability as $\delta \to 0$. By definition of ϕ^δ,ϕ^o we have

$$\phi^\delta(s,x) = J\ (s,x;\ Y^{\delta*}) \le J^\delta(s,x;\ Y^{o*}),$$
$$\phi^o(s,x) = J^o(s,x;\ Y^{o*}) \le J^o(s,x;\ Y^{\delta*}).$$

These imply

$$J^\delta(s,x;\ Y^{\delta*}) - J^o(s,x;\ Y^{\delta*}) \le \phi^\delta - \phi^o \le J^\delta(s,x;\ Y^{o*}) - J^o(s,x;\ Y^{o*}),$$

i.e.,

$$E_{s,x}\int_s^T [\xi^{\delta *'}(t)M(t)\xi^{\delta *}(t) - \zeta'(t)M(t)\zeta(t)]dt \le \phi^\delta - \phi^o$$

$$\le E_{s,x}\int_s^T [\xi'(t)M(t)\xi(t) - \xi^{o*'}(t)M(t)\xi^{o*}(t)]dt.$$

Since $E_{s,x}(\xi^{\delta *'}M\xi^{\delta *} - \zeta'M\zeta)^2$ and $E_{s,x}(\xi'M\xi - \xi^{o*'}M\xi^{o*})^2$ are bounded and the bounds do not depend on δ, we use Lebesgue dominated convergence theorem to get the result. Q.E.D.

Lemma 4.2. $\phi_x^\delta(s,x) \to \phi_x^o(s,x)$ uniformly on any compact set.

Proof. Since in (3.7) ν is independent of δ, (3.7) implies that ϕ_x^δ is uniformly bounded on any compact set. By Theorem 3.1, $\phi^\delta \in \mathcal{F}_0$. Moreover, we know $\phi^\delta \in C^\infty$. Hence ϕ_x^δ is equicontinuous on any compact set. By Ascoli's theorem, there exists a subsequence $\phi_x^{\delta_n}$ which converges uniformly to a limit ζ. Let us show that $\zeta = \phi_x^o$. Since

$$\int_{x_{oi}}^{x_i} \phi_{x_i}^{\delta_n} dx_i \to \int_{x_{oi}}^{x_i} \zeta_i dx_i$$

and using Lemma 4.1

$$\int_{x_{oi}}^{x_i} \phi_{x_i}^{\delta_n} dx_i = \phi^{\delta_n}(s,x_i) - \phi^{\delta_n}(s,x_{oi}) \to \phi^o(s,x_i) - \phi^o(s,x_{oi}).$$

Then using fundamental of calculus, we have $\zeta_{x_i} = \phi_{x_i}^o$ and hence the lemma. Q.E.D.

Lemma 4.3. $\xi^{\delta*} \to \xi^{o*}$ in probability as $\delta \to 0$.

Proof. From (1.1^{δ}), (1.1^{o}) we have

$$d(\xi^{\delta*}-\xi^{o*}) = (A(\xi^{\delta*}-\xi^{o*})+\delta g(\xi^{\delta*})+B(Y^{\delta*}(t,\xi^{\delta*})-Y^{o*}(t,\xi^{\delta*}) + Y^{o*}(t,\xi^{\delta*})-Y^{o*}(t,\xi^{o*})))dt$$

with $\xi^{\delta*}(s) - \xi^{o*}(s) = 0$. By (1.5)

$$Y^{o*}(t,\xi^{\delta*}) - Y^{o*}(t,\xi^{o*}) = N^{-1}(t)B'(t)K(t)(\xi^{\delta*}(t) - \xi^{o*}(t)).$$

Let $X(t,u)$ be the principal matrix solution at initial time v of

$$dX = (A(t) - N^{-1}(t)B'(t)K(t))X\,dt.$$

Then

$$\xi^{\delta*} - \xi^{o*} = \int_s^t X(t,v)(\delta g(\xi^{\delta*}(v)) + B(Y^{\delta*}(v,\xi^{\delta*}(v)) - Y^{o*}(v,\xi^{\delta*}(v))))dv.$$

Since $E_{s,x}|g(\xi^{\delta*}(v))|$, $E_{s,x}|Y^{\delta*}(v,\xi^{\delta*}(v)) - Y^{o*}(v,\xi^{\delta*}(v))|^2$ are bounded independent of δ and $Y^{\delta*}(v,\xi^{\delta*}(v))$ approaches $Y^{o*}(v,\xi^{\delta*}(v))$ almost surely,

$$E|\xi^{\delta*} - \xi^{o*}| \to 0 \text{ as } \delta \to 0.$$

Thus

$$\zeta^{\delta *} \to \xi^{o *} \quad \text{in probability as} \quad \delta \to 0. \qquad \text{Q.E.D.}$$

We now consider formula (1.11) with $k = 1$.

Lemma 4.4. $\phi^\delta = \phi^o + \delta\theta_1 + o(\delta)$ where $\delta^{-1}o(\delta) \to 0$ as $\delta \to 0$ uniformly on any compact set and θ_1 satisfies

$$(\theta_1)_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'(\theta_1)_{xx}\} + (\theta_1)_x \cdot f^{o,Y^{o*}} + \phi_x^o \cdot g = 0 \qquad (4.3^o)$$

with the initial data $\theta_1(T,x) = 0$.

Proof. We can show $\theta_1(s,x)$ has following form

$$\theta_1(s,x) = E_{s,x}\int_s^T \phi_x^o(u,\xi^{o*}(t))\cdot g(\xi^{o*}(t))dt.$$

For $\delta > 0$, let

$$\theta_1^\delta = \delta^{-1}(\phi^\delta - \phi^o).$$

By (1.7^δ) and (1.7^o), θ_1^δ satisfies

$$(\theta_1^\delta)_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'(\theta_1^\delta)_{xx}\} + (\theta_1^\delta)_x \cdot f^{\delta,\frac{Y^{\delta *}+Y^{o*}}{2}} + \phi_x^o \cdot g = 0. \qquad (4.3^\delta)$$

Let ζ^δ be the solution of

$$d\zeta^\delta(t) = f^{\delta,\frac{Y^{\delta *}+Y^{o*}}{2}}(t,\zeta^\delta)dt + \sigma(t)dw$$

with initial data $\zeta^\delta(s) = x$. Since $\frac{1}{2}(Y^{\delta *} + Y^{o*}) \in \mathcal{Y}$, the solution ζ^δ exists which is unique in the usual sense and all of its moments are bounded. Similar to the proof of Lemma 4.3, we can show that $\zeta^\delta \to \xi^{o*}$ in probability as $\delta \to 0$. Also by Lemma 2.3,

$$\theta_1^\delta(s,x) = E_{s,x}\int_s^T \phi_x^o(t,\zeta^\delta(t))\cdot g(\zeta^\delta(t))dt$$

where $\phi_x \cdot g$ satisfies polynomial growth condition, i.e., $E_{s,x}|\phi_x^o(t,\zeta^\delta(t))\cdot g(\zeta^\delta(t))|^2$ is bounded independent of δ. Hence

$$\lim_{\delta\to 0} E_{s,x}\int_s^T \phi_x^o(t,\zeta^\delta(t))\cdot g(\zeta^\delta(t))dt$$

$$= E_{s,x}\int_s^T \phi_x^o(t,\xi^o(t))\cdot g(\xi^o(t))dt.$$

Thus $\theta_1^\delta \to \theta_1$ as $\delta \to 0$. The convergence is uniformly on any compact set. Hence the lemma is proved. Q.E.D..

Now we consider formula (1.12) with $k = 1$.

Lemma 4.5. $\phi_x^\delta = \phi_x^o + \delta(\theta_1)_x + o(\delta)$ where $\delta^{-1}o(\delta) \to 0$ as $\delta \to 0$ uniformly on any compact set.

Proof. It is equivalent to show $(\theta_1^\delta)_x \to (\theta_1)_x$ as $\delta \to 0$. Using (4.3^δ), $(\theta_1^\delta)_{x_i}$ satisfies

$$(\theta_{1x_i}^\delta)_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'(\theta_{1x_i}^\delta)_{xx} + \\ + (\theta_{1x_i}^\delta)_x\cdot f^{\delta,Y^{\delta*}} + (\theta_1^\delta)_x\cdot f_{x_i}^{\delta,Y^{o*}} + (\phi_x^o\cdot g)_{x_i} = 0 \tag{4.5^δ}$$

with $\theta^{\delta}_{1x_i}(T,x) = 0$. Let W^{δ} be the principal matrix solution at initial time s of

$$dW^{\delta} = f^{\delta,Y^{o*}}(t,\xi^{\delta*}(t))W^{\delta}dt.$$

In fact, $f_x^{\delta,Y^{o*}} = A + \delta g_x - BN^{-1}B'K$. Using the similar technique of proof boundedness of W we can show that W^{δ} is bounded and the bound does not depend on x. Similar as in Lemma 3.2 we can show

$$(\theta^{\delta}_1)_x(s,x) = E_{s,x}\int_s^T (\phi^o_x\cdot g)_x W^{\delta}\Big|_{(t,\xi^{\delta*}(t))} dt. \qquad (4.6^{\delta})$$

Since moments of $\xi^{\delta*}(t)$ are bounded and $(\phi^o_x\cdot g)_x W^{\delta} \in C_p$, then

$$\lim_{\delta\to 0}(\theta^{\delta}_1)_x(s,x) = E_{s,x}\int_s^T (\phi^o_x\cdot g)_x W^o\Big|_{t,\xi^{o*}(t)} dt \qquad (4.7)$$

where W^o is the principal matrix solution at initial time s of

$$dW^o = (A - BN^{-1}B'K)W^o dt.$$

It is easy to see the right hand side of (4.7) is just $(\theta_1)_x(s,x)$. Indeed, from (4.3^o) we have

$$(\theta_{1x_i})_s + \frac{1}{2}\,\mathrm{tr}\{\sigma\sigma'(\theta_{1x_i})_{xx}\} + (\theta_{1x_i})_x\cdot f^{o,Y^{o*}} + (\theta_1)_x\cdot f^{o,Y^{o*}}_{x_i}$$
$$+ (\phi^o_x\cdot g)_{x_i} = 0.$$

By a similar procedure

$$(\theta_1)_x(s,x) = E_{s,x}\int_s^T (\phi_x^o\cdot g)_x W^o\Big|_{t,\xi^{o*}(t))} dt.$$

Hence $(\theta_1^\delta)_x \to (\theta_1)_x$ as $\delta \to 0$ uniformly on any compact set. Q.E.D.

Lemma 4.6. $\phi^\delta = \phi^o + \delta\theta_1 + \delta^2\theta_2 + o(\delta^2)$ where $\delta^{-2}o(\delta^2) \to 0$ as $\delta \to 0$ uniformly on any compact set and θ_2 is defined by

$$(\theta_2)_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'(\theta_2)_{xx}\} + (\theta_2)_x\cdot f^{0,Y^{o*}} + \frac{1}{2}\theta_{1x}H_{pp}\theta_{1x} \qquad (4.8^o)$$
$$+ (\theta_1)_x\cdot g = 0$$

with initial data $\theta_2(T,x) = 0$.

Proof. Let

$$\theta_2^\delta = \delta^{-1}(\theta_1^\delta-\theta_1) = \delta^{-2}(\phi^\delta-\phi^o-\delta\theta_1).$$

Then the problem is equivalent to $\theta_2^\delta \to \theta_2$ as $\delta \to 0$. By (4.3^δ), (4.3^o), θ_2^δ satisfies

$$(\theta_2^\delta)_s + \frac{1}{2}\operatorname{tr}\{\sigma\sigma'(\theta_2^\delta)_{xx}\} + (\theta_2^\delta)_x\cdot f^{\delta,\frac{Y^{\delta*}+Y^{o*}}{2}} \qquad (4.8^\delta)$$
$$+ \frac{1}{2}(\theta_1)_x H_{pp}^o(\theta_1^\delta)_x + (\theta_1)_x\cdot g = 0.$$

Using Lemma 2.3,

$$\theta_2^\delta(s,x) = E_{s,x}\int_s^T (\tfrac{1}{2}\,\theta_{1x}(t,\zeta^\delta(t))H^o_{pp}\theta^\delta_{1x}(t,\zeta^\delta(t)) + \theta_{1x}(t,\xi^\delta(t))\cdot g(\zeta^\delta(t)))dt. \qquad (4.9^\delta)$$

Since $\zeta^\delta \to \xi^{o*}$ in probability as $\delta \to 0$ and by (4.6^o), (4.6^δ) we can see that both $(\theta_1)_x$ and $(\theta_1^\delta)_x$ belong to C_p, i.e., the integrand of (4.9^δ) also beongs to C_p, hence

$$\lim_{\delta\to 0}\theta_2^\delta(s,x) = E_{s,x}\int_s^T [\tfrac{1}{2}\theta_{1x}(t),\xi^{o*}(t))H^o_{pp}\theta_{1x}(t,\xi^{o*}(t)) + \theta_{1x}(t,\xi^{o*}(t))\cdot g(\xi^{o*}(t))]dt$$

and the right hand side is just $\theta_2(s,x)$. In fact from (4.8^o)

$$\theta_2(s,x) = E_{s,x}\int_s^T (\tfrac{1}{2}\,\theta_{1x}(t,\xi^{o*}(t))H^o_{pp}\theta_{1x}(t,\,\xi^{o*}(t)) + \theta_{1x}(t,\xi^{o*}(t))\cdot g(\xi^{o*}(t)))dt. \qquad (4.9^o)$$

Therefore, $\theta_2^\delta \to \theta_2$ as $\delta \to 0$ uniformly on any compact set. Q.E.D.

We can continue the procedure and finally we have

<u>Theorem 4.1</u>. The expansions of (1.11), (1.12) are valid for any k to ℓ and hold uniformly on any compact set.

<u>Corollary 4.1</u>. $Y^{\delta *}(s,x) = Y^{o*}(s,x) - \frac{\delta}{2}N^{-1}(s)B'(s)(\theta_1)_x'(s,x) - \cdots - \frac{\delta^k}{2}N^{-1}(s)B'(s)(\theta_k)_x'(s,x) + o(\delta^k)$ where $k \le \ell$.

<u>Proof</u>. Use (2.2) and Theorem 4.1. Q.E.D.

Now we consider goodness of $Y^{o*}(s,x)$ in perturbed problem. By Corollary 4.1 we know Y^{o*} gives approximately the optimal control policy in the perturbed problem for small δ. It is also plausible that Y^{o*} should give approximately the optimum in the perturbed problem. The above lemmas and their method of proof put this rough statement on a quantitative basis. Let

$$\Phi^{\delta}(s,x) = J^{\delta}(s,x;\ Y^{o*}).$$

In particular, $\Phi^{o}(s,x) = \phi^{o}(s,x)$. For $\delta > 0$, $\Phi^{\delta}(s,x) - \phi^{\delta}(s,x)$ represents how much Y^{o*} fails to be optimal in the perturbed problem. It is known that $\Phi^{\delta} \in \mathcal{F}_o$ and satisfies the linear parabolic equation

$$(\Phi^{\delta})_s + \frac{1}{2}\,\mathrm{tr}(\sigma\sigma'\Phi^{\delta}_{xx}) + \Phi^{\delta}_x\cdot f^{\delta,Y^{o*}} + x'Mx + Y^{o*\prime}NY^{o*} = 0 \quad (4.10)$$

with initial data $\Phi^{\delta}(T,x) = 0$. Let us write

$$\Phi^{\delta} = \phi^{o} + \delta\chi_1 + \delta^2\chi_2 + o(\delta^2). \quad (4.11)$$

By the same procedure as before, we have for $k = 1,2$

$$(\chi_k)_s + \frac{1}{2}\,\mathrm{tr}(\sigma\sigma'(\chi_k)_{xx}) + (\chi_k)_x\cdot f^{o,Y^{o*}} + (\chi_{k-1})_x\cdot g = 0 \quad (4.12)$$

where $\chi_o = \phi^{o}$. Let $\chi_1^{\delta} = \delta^{-1}(\Phi^{\delta}-\phi^{o})$, $\chi_2^{\delta} = \delta^{-2}(\Phi^{\delta}-\phi^{o}-\delta\chi_1)$.

Hence for $k = 1,2$

$$(\chi_k^\delta)_s + \frac{1}{2}\operatorname{tr}(\sigma\sigma'(\chi_k^\delta)_{xx}) + (\chi_k^\delta)_x \cdot f^{\delta,Y^{o*}} + (\chi_{k-1})_x \cdot g = 0.$$

Then

$$\chi_1 = \theta_1,$$

$$\chi_2 = E_{s,x}\int_s^T \theta_{1x}(t,\xi^o(t))\cdot g(\xi^o(t))dt.$$

By the same procedure, we can prove $\chi_1^\delta \to \chi_1$, $\chi_2^\delta \to \chi_2$ as $\delta \to 0$ uniformly on any compact set. By comparing with Lemma 4.6, we find that

$$\Phi^\delta(s,x) - \phi^\delta(s,x)$$
$$= -\frac{1}{2}\delta^2 E_{s,x}\int_s^T \theta_{1x}(t,\xi^o(t))H_{pp}^o\theta_{1x}(t,\xi^o(t))dt + o(\delta^2). \tag{4.13}$$

Formula (4.13) shows that Y^{o*} gives within order the square of the intensity of perturbation δ of the optimum.

Example. Let ξ^δ be the solution of the scalar Ito equation

$$d\xi^\delta = (-\delta(\xi^\delta)^3 + u(t))dt + \sigma dw$$

with $\xi^\delta(x) = x$. The control set is R. The criterion of performance is

$$J(s,x;\ u) = E_{s,x} \int_s^1 (\xi^{\delta 2} + u^2)dt.$$

By [FR1, Section 6.5], we have

$$\phi^o(s,x) = K(s)x^2 + q(s)$$

where $K(s) = \tanh(T-s)$ and $q(s) = \sigma^2 \ell n \cosh(T-s)$. By Theorem 4.1

$$\theta_1(s,x) = -2 \int_s^T k(t) E_{s,x} \xi^{o4}(t)dt$$

$$= -2(\beta_1(s)x^4 + \beta_2(s)x^2 + \beta_3(s))$$

where

$$\beta_1(s) = \frac{1}{4}\left(1 - \frac{1}{\cosh^4(T-s)}\right)$$

$$\beta_2(s) = \frac{6\sigma^2}{4\cosh^2(T-s)} (\tanh(T-s)(\cosh^4(T-s) - 1)$$

$$- \frac{1}{8} \sin 4(T-s) + \frac{T-s}{2})$$

$$\beta_3(s) = 6\sigma^4 (\tanh^2(T-s)(\cosh^4(T-s) - 1)$$

$$- \tanh(T-s)\left(\frac{1}{8} \sin^4(T-s) - \frac{T-s}{2}\right)$$

$$+ \frac{1}{8} (1-\cosh^2(T-s) + \frac{1}{2\cosh^2(T-s)} (\cosh^4(T-s)-1)) .$$

Then

$$\phi^{\delta}(s,x) = \phi^{o}(s,x) + \delta\theta_1(s,x) + o(\delta)$$

$$Y^{\delta *}(s,x) = Y^{o*}(s,x) + \delta(4\beta_1(s)x^3 + 2\beta_2(s)x) + o(\delta).$$

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PERTURBED STOCHASTIC LINEAR REGULATOR PROBLEMS

by

Chun-Ping Tsai

The theory of optimal feedback control of Markov diffusion processes has been well developed. However, it is a difficult matter to calculate optimal feedback controls, except for the linear regulator problem and some other special cases.

In this paper a nonlinear perturbation of the stochastic linear regulator is considered. An algorithm is given for computing approximately the optimal feedback control, if the nonlinearity appearing in the nonlinear stochastic differential equations governing the system is a polynomial in the state variables. Under appropriate assumptions on the nonlinearity, the method is justified in a mathematically rigorous way. The quantities which need to be computed to find the optimum approximately can be expressed in terms of higher order moments of a known gaussian process, namely the state process for the optimum linear regulator.

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This paper is concerned with the approximate solution of stochastic optimal control problems which arise by perturbing the stochastic linear regulator problem, through an additive term with a small parameter δ in the drift coefficient of the unperturbed dynamical equations. The system states are assumed completely observable. Our main results concern expansions of solutions of the perturbed equation in powers $\delta, \delta^2, \delta^3, \ldots$ of the small parameter δ. delta

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